Exchange economy with bliss-point and linear utility functions

In an economy there are two consumers, A and B, who are endowed with two commodities, x and y. The preferences of these consumers are given by the following utility functions:

$$u_A = 1 - (x_A - 4)^2 - (y_A - 3)^2$$

$$u_B = x_B + y_B$$

Consumer A has an initial endowment $\omega_A = (5, 0)$, while consumer B has an initial endowment $\omega_B = (0, 5)$. Assuming that consumers take prices as given, answer the following:

1. What relative price equilibrates the market? What are the consumption bundles of both consumers after trade?
2. Suppose now that the government decides to "redistribute" wealth by taking 1 unit of good x from consumer A and transferring it to consumer B, and taking 1 unit of good y from consumer B and transferring it to consumer A. Does the equilibrium change?
3. What conditions must hold to ensure that we are in a market equilibrium (Walrasian equilibrium)?
4. Find the contract curve and the Utility Possibility Frontier (UPF)
5. Verify that the equilibrium found in part (1) is efficient, using your result from part (4)

Solution

1. Let prices be (p_x, p_y), with $p_x > 0$ and $p_y > 0$. Consumer A has income

$$m_A = 5p_x$$

and consumer B has income

$$m_B = 5p_y$$

The total endowment in the economy is

$$\bar{x} = 5 \qquad \bar{y} = 5$$

Consumer A

Consumer A solves

$$\max_{x_A, y_A} \; 1 - (x_A - 4)^2 - (y_A - 3)^2$$

subject to

$$p_x x_A + p_y y_A \leq 5p_x$$

The bliss point of consumer A is

$$(4, 3)$$

This bundle is affordable if

$$4p_x + 3p_y \leq 5p_x$$

$$3p_y \leq p_x$$

$$\frac{p_x}{p_y} \geq 3$$

Hence,

$$(x_A, y_A) = (4, 3) \quad \text{if} \quad \frac{p_x}{p_y} \geq 3$$

If instead

$$\frac{p_x}{p_y} < 3$$

then the bliss point is not affordable, so the budget constraint binds and consumer A solves

$$\max_{x_A, y_A} \; 1 - (x_A - 4)^2 - (y_A - 3)^2$$

subject to

$$p_x x_A + p_y y_A = 5p_x$$

The Lagrangian is

$$\mathcal{L}_A = 1 - (x_A - 4)^2 - (y_A - 3)^2 + \lambda_A(5p_x - p_x x_A - p_y y_A)$$

The first-order conditions are

$$-2(x_A - 4) - \lambda_A p_x = 0$$

$$-2(y_A - 3) - \lambda_A p_y = 0$$

$$5p_x - p_x x_A - p_y y_A = 0$$

We will use these conditions below at the equilibrium price

Consumer B

Consumer B solves

$$\max_{x_B, y_B} \; x_B + y_B$$

subject to

$$p_x x_B + p_y y_B \leq 5p_y$$

Since consumer B values both goods equally in utility terms, demand depends on which good is cheaper
If

$$p_x < p_y$$

then x is cheaper, so consumer B spends all income on x

$$(x_B, y_B) = \left(\frac{5p_y}{p_x}, 0\right)$$

If

$$p_x > p_y$$

then y is cheaper, so consumer B spends all income on y

$$(x_B, y_B) = (0, 5)$$

If

$$p_x = p_y$$

then consumer B is indifferent among all bundles satisfying

$$x_B + y_B = 5$$

Equilibrium price

We now check which relative price can clear markets

If

$$p_x < p_y$$

then consumer B demands

$$x_B = \frac{5p_y}{p_x} > 5$$

which is impossible, since total endowment of good x is only 5
So we cannot have

$$\frac{p_x}{p_y} < 1$$

If

$$p_x > p_y$$

then consumer B demands

$$y_B = 5$$

leaving

$$y_A = 0$$

But consumer A's optimum always has positive y_A, so this cannot be an equilibrium
So we cannot have

$$\frac{p_x}{p_y} > 1$$

Therefore, the only possible equilibrium price ratio is

$$\frac{p_x}{p_y} = 1$$

Equilibrium allocation
Normalize prices as

$$p_x = p_y = 1$$

Then consumer A's problem becomes

$$\max_{x_A, y_A} \; 1 - (x_A - 4)^2 - (y_A - 3)^2$$

subject to

$$x_A + y_A = 5$$

The Lagrangian is

$$\mathcal{L}_A = 1 - (x_A - 4)^2 - (y_A - 3)^2 + \lambda_A(5 - x_A - y_A)$$

The first-order conditions are

$$-2(x_A - 4) - \lambda_A = 0$$

$$-2(y_A - 3) - \lambda_A = 0$$

$$5 - x_A - y_A = 0$$

From the first two equations,

$$x_A - 4 = y_A - 3$$

$$y_A = x_A - 1$$

Using the budget constraint,

$$x_A + (x_A - 1) = 5$$

$$2x_A = 6$$

$$x_A = 3$$

$$y_A = 2$$

Thus consumer A's demand is

$$(x_A, y_A) = (3, 2)$$

Since total endowments are $(5, 5)$, market clearing implies consumer B must consume

$$x_B = 5 - 3 = 2$$

$$y_B = 5 - 2 = 3$$

So

$$(x_B, y_B) = (2, 3)$$

This bundle satisfies

$$x_B + y_B = 5$$

so it belongs to consumer B's demand correspondence when $p_x = p_y$
Hence, the equilibrium allocation is

$$(x_A, y_A) = (3, 2) \qquad (x_B, y_B) = (2, 3)$$

Therefore, the market-clearing relative price is $\frac{p_x}{p_y} = 1$, and the equilibrium consumption bundles are $(3, 2)$ for consumer A and $(2, 3)$ for consumer B

2. After the redistribution, the new endowments are

$$\omega'_A = (4, 1) \qquad \omega'_B = (1, 4)$$

The total endowment in the economy remains

$$\bar{x} = 5 \qquad \bar{y} = 5$$

Let prices be (p_x, p_y), with $p_x > 0$ and $p_y > 0$

Consumer A

Consumer A's income is now

$$m_A = 4p_x + p_y$$

Consumer A solves

$$\max_{x_A, y_A} \; 1 - (x_A - 4)^2 - (y_A - 3)^2$$

subject to

$$p_x x_A + p_y y_A \leq 4p_x + p_y$$

Notice that the bliss point $(4, 3)$ is never affordable, because its cost is

$$4p_x + 3p_y > 4p_x + p_y$$

Thus, the budget constraint binds and consumer A solves

$$\max_{x_A, y_A} \; 1 - (x_A - 4)^2 - (y_A - 3)^2$$

subject to

$$p_x x_A + p_y y_A = 4p_x + p_y$$

The Lagrangian is

$$\mathcal{L}_A = 1 - (x_A - 4)^2 - (y_A - 3)^2 + \lambda_A(4p_x + p_y - p_x x_A - p_y y_A)$$

The first-order conditions are

$$-2(x_A - 4) - \lambda_A p_x = 0$$

$$-2(y_A - 3) - \lambda_A p_y = 0$$

$$4p_x + p_y - p_x x_A - p_y y_A = 0$$

From the first two equations,

$$x_A = 4 - \frac{\lambda_A p_x}{2} \qquad y_A = 3 - \frac{\lambda_A p_y}{2}$$

Substituting into the budget constraint,

$$p_x\left(4-\frac{\lambda_A p_x}{2}\right)+p_y\left(3-\frac{\lambda_A p_y}{2}\right)=4p_x+p_y$$

$$4p_x+3p_y-\frac{\lambda_A}{2}(p_x^2+p_y^2)=4p_x+p_y$$

$$2p_y=\frac{\lambda_A}{2}(p_x^2+p_y^2)$$

$$\lambda_A=\frac{4p_y}{p_x^2+p_y^2}$$

Therefore, consumer A's demand is

$$x_A=4-\frac{2p_xp_y}{p_x^2+p_y^2}$$

$$y_A=3-\frac{2p_y^2}{p_x^2+p_y^2}$$

Consumer B
Consumer B's income is

$$m_B=p_x+4p_y$$

Consumer B solves

$$\max_{x_B,y_B}\ x_B+y_B$$

subject to

$$p_xx_B+p_yy_B\leq p_x+4p_y$$

As before, consumer B's demand depends on which good is cheaper
If

$$p_x<p_y$$

then consumer B buys only good x

$$(x_B,y_B)=\left(\frac{p_x+4p_y}{p_x},0\right)$$

But then

$$x_B=1+4\frac{p_y}{p_x}>5$$

which is impossible, since total endowment of good x is only 5
So this cannot be an equilibrium
If

$$p_x > p_y$$

then consumer B buys only good y

$$(x_B, y_B) = \left(0, \frac{p_x + 4p_y}{p_y}\right)$$

But then

$$y_B = \frac{p_x}{p_y} + 4 > 5$$

which is impossible, since total endowment of good y is only 5
So this cannot be an equilibrium either
Therefore, the only possible equilibrium relative price is

$$\frac{p_x}{p_y} = 1$$

Equilibrium allocation
Normalize prices as

$$p_x = p_y = 1$$

Then consumer A's budget becomes

$$x_A + y_A = 5$$

and the demand formulas become

$$x_A = 4 - \frac{2}{2} = 3$$

$$y_A = 3 - \frac{2}{2} = 2$$

Thus,

$$(x_A, y_A) = (3, 2)$$

Since total endowments are still $(5, 5)$, market clearing implies

$$x_B = 5 - 3 = 2 \qquad y_B = 5 - 2 = 3$$

Hence,

$$(x_B, y_B) = (2, 3)$$

This bundle satisfies

$$x_B + y_B = 5$$

so it belongs to consumer B's demand correspondence when $p_x = p_y$
Therefore, the new equilibrium allocation is

$$(x_A, y_A) = (3, 2) \qquad (x_B, y_B) = (2, 3)$$

The equilibrium does not change. After the redistribution, the market-clearing relative price remains $\frac{p_x}{p_y} = 1$, and the equilibrium allocation is still $(3, 2)$ for consumer A and $(2, 3)$ for consumer B

3. A Walrasian equilibrium in this economy is a price vector (p_x, p_y) with $p_x > 0$ and $p_y > 0$, together with an allocation

$$(x_A, y_A), (x_B, y_B)$$

such that the following conditions hold

(a) **Utility maximization**
Given prices, each consumer chooses a utility-maximizing bundle from their budget set
Consumer A solves

$$(x_A, y_A) \in \arg \max_{x_A \geq 0,\ y_A \geq 0} \left\{1 - (x_A - 4)^2 - (y_A - 3)^2 \ :\ p_x x_A + p_y y_A \leq 5p_x\right\}$$

Consumer B solves

$$(x_B, y_B) \in \arg \max_{x_B \geq 0,\ y_B \geq 0} \{x_B + y_B \ :\ p_x x_B + p_y y_B \leq 5p_y\}$$

(b) **Feasibility and market clearing**
Total consumption must equal total endowments

$$x_A + x_B = 5$$

$$y_A + y_B = 5$$

(c) **Positive prices**
Prices must satisfy

$$p_x > 0 \qquad p_y > 0$$

We can now quickly verify that the equilibrium found in part (1),

$$\frac{p_x}{p_y} = 1 \qquad (x_A, y_A) = (3, 2) \qquad (x_B, y_B) = (2, 3)$$

satisfies all these conditions

Normalize prices as

$$p_x = p_y = 1$$

Then markets clear because

$$x_A + x_B = 3 + 2 = 5$$

$$y_A + y_B = 2 + 3 = 5$$

Consumer A's bundle is affordable and exhausts income

$$3 + 2 = 5 = 5p_x$$

Consumer B's bundle is also affordable and exhausts income

$$2 + 3 = 5 = 5p_y$$

At these prices, consumer A chooses the point on the budget line $x_A + y_A = 5$ that is closest to the bliss point $(4, 3)$, which is $(3, 2)$

Consumer B has linear utility $x_B + y_B$, and when $p_x = p_y$ every bundle satisfying

$$x_B + y_B = 5$$

is optimal, so $(2, 3)$ belongs to consumer B's demand correspondence

Hence, the allocation found in part (1), together with the relative price $\frac{p_x}{p_y} = 1$, satisfies all the conditions of a Walrasian equilibrium

4. To find the contract curve, we characterize the Pareto efficient allocations

   Using feasibility,

$$x_B = 5 - x_A \qquad y_B = 5 - y_A$$

   Consumer B's utility is

$$u_B = x_B + y_B = 10 - (x_A + y_A)$$

   Thus, for a given utility level of consumer A, Pareto efficiency requires choosing the bundle of consumer A that achieves that utility with the smallest possible value of $x_A + y_A$

   For interior efficient allocations, the slope condition is

$$MRS_A = MRS_B$$

   For consumer A,

$$MU_{x_A} = -2(x_A - 4) \qquad MU_{y_A} = -2(y_A - 3)$$

   so

$$MRS_A = \frac{MU_{x_A}}{MU_{y_A}} = \frac{x_A - 4}{y_A - 3}$$

   For consumer B,

$$MU_{x_B} = 1 \qquad MU_{y_B} = 1$$

   so

$$MRS_B = 1$$

   Hence, interior efficiency requires

$$\frac{x_A - 4}{y_A - 3} = 1$$

$$x_A - y_A = 1$$

   This gives the interior part of the contract curve. Since the bliss point of consumer A is $(4, 3)$, only the southwest part of this line can be efficient, because allocations northeast of $(4, 3)$ are Pareto dominated by moving back toward the bliss point and giving the released goods to consumer B

   Therefore, the interior efficient segment is

$$y_A = x_A - 1 \qquad 1 \leq x_A \leq 4$$

   For lower utility levels of consumer A, the efficient allocation lies on the boundary $y_A = 0$, because this minimizes $x_A + y_A$ among bundles that deliver that utility

   Hence, the boundary efficient segment is

$$y_A = 0 \qquad 0 \leq x_A \leq 1$$

Combining both parts, the contract curve is

$$\mathcal{C} = \{(x_A, y_A) : y_A = 0,\ 0 \le x_A \le 1\} \cup \{(x_A, y_A) : y_A = x_A - 1,\ 1 \le x_A \le 4\}$$

We now construct the Utility Possibility Frontier. The UPF is the set of Pareto efficient utility pairs and is usually written as consumer B's utility as a function of consumer A's utility

First segment of the contract curve: $y_A = 0$**,** $0 \le x_A \le 1$

Along this segment,

$$U_A = 1 - (x_A - 4)^2 - 9$$

$$U_A = -x_A^2 + 8x_A - 24$$

and

$$U_B = 10 - x_A$$

From

$$U_A = -(x_A - 4)^2 - 8$$

we obtain

$$x_A = 4 - \sqrt{-U_A - 8}$$

Thus,

$$U_B(U_A) = 6 + \sqrt{-U_A - 8} \qquad -24 \le U_A \le -17$$

Second segment of the contract curve: $y_A = x_A - 1$**,** $1 \le x_A \le 4$

Along this segment,

$$U_A = 1 - (x_A - 4)^2 - (x_A - 4)^2$$

$$U_A = 1 - 2(x_A - 4)^2$$

and

$$U_B = 10 - (x_A + (x_A - 1)) = 11 - 2x_A$$

From

$$U_A = 1 - 2(x_A - 4)^2$$

and using $1 \le x_A \le 4$, we get

$$x_A = 4 - \sqrt{\frac{1 - U_A}{2}}$$

Therefore,

$$U_B(U_A) = 3 + \sqrt{2(1 - U_A)} \qquad -17 \le U_A \le 1$$

So the Utility Possibility Frontier is

$$U_B(U_A) = \begin{cases} 6 + \sqrt{-U_A - 8} & \text{if } -24 \le U_A \le -17 \\ 3 + \sqrt{2(1 - U_A)} & \text{if } -17 \le U_A \le 1 \end{cases}$$

Hence, the contract curve is the union of the segment $y_A = 0$ for $0 \le x_A \le 1$ and the segment $y_A = x_A - 1$ for $1 \le x_A \le 4$, and the UPF is the piecewise function above

5. From part (1), the equilibrium allocation is

$$(x_A, y_A) = (3, 2) \qquad (x_B, y_B) = (2, 3)$$

From part (4), one part of the contract curve is

$$y_A = x_A - 1 \qquad 1 \leq x_A \leq 4$$

At the equilibrium allocation,

$$2 = 3 - 1$$

so $(3, 2)$ belongs to the contract curve

Therefore, the equilibrium allocation is Pareto efficient

We can also verify this using the Utility Possibility Frontier

At the equilibrium allocation,

$$U_A = 1 - (3 - 4)^2 - (2 - 3)^2$$

$$U_A = -1$$

and

$$U_B = 2 + 3 = 5$$

Since $-1 \in [-17, 1]$, we use the second branch of the UPF

$$U_B(U_A) = 3 + \sqrt{2(1 - U_A)}$$

Substituting $U_A = -1$,

$$U_B = 3 + \sqrt{2(1 - (-1))}$$

$$U_B = 3 + \sqrt{4}$$

$$U_B = 5$$

which is exactly the utility of consumer B at the equilibrium allocation

Hence, the equilibrium found in part (1) is efficient because it lies on the contract curve, and equivalently because the associated utility pair lies on the Utility Possibility Frontier